A New Technique for Increasing the Flexibility of Recursive Least Squares Data Smoothing

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A method of performing recursive least squares data smoothing is described in which optimum (or arbitrary) weights can be assigned to the observations. The usual restriction of a constant data interval can be removed without affecting the optimum weighting or recursive features. The method also provides an instantaneous (i.e. real time) estimate of the statistical accuracy in the smoothed coordinates for a set of arbitrary data intervals. Optimum gate sizes for arbitrary predictions can be determined. These features greatly increase the flexibility of recursive least squares data smoothing, and several applications are discussed.

I. INTRODUCTION

During the past few years, a need has arisen for data smoothing techniques which can be applied, in real time, to radar observations of bodies traveling along highly predictable trajectories. The observations are usually processed in digital computers, so that much effort has been expended in devising techniques suited to the advantages and limitations of computers. This paper is concerned with one such technique, recursive least squares smoothing, whose theoretical foundation was established several years ago.[1] It is our purpose to show how this technique can be made considerably more flexible so as to encompass a wide variety of practical situations while maintaining its suitability for computer use.

By the term "recursive," we mean that a smoothed coordinate is determined from a previously computed average of past data (one number) and a new observation. Thus the storage requirements are independent of the number of observations and are actually quite modest. We will also use the term "optimum smoothing," which is to be interpreted in the least squares sense, i.e., the weighting of data inversely proportional to

their error variances. The ability to perform optimum smoothing in real time where the variations (not necessarily the magnitudes) in observational error are determinable (e.g., trajectory-dependent errors) is one of the advantages of the method described here. Over long periods of observation, a significant increase in the accuracy of the smoothed coordinates may be achieved by properly weighting the data. In the actual derivation, however, no restrictions are placed on the weighting factors, thus allowing an arbitrary sequence of weights to be placed on the data.

Sections II and III are devoted to the estimation of position and velocity assuming the body under observation is traveling along a straight line with no acceleration. In Section II we consider the case of constant data intervals where some emphasis has been placed on the ability to compensate properly for missing observations. We have also included formulas for determining optimum gate sizes for predicted observations, which are of considerable importance in some tracking systems. In Section III the method is extended to handle the case of arbitrary data intervals.

That the restriction to linear flight does not limit the applicability of this method to many practical cases is shown in Section IV, where methods for including the effects of known accelerations are discussed. In Appendix A we show that the sum of square deviations from the least squares line of regression may be obtained recursively. This result may be useful as a means of real-time error detection or as a way of providing an instantaneous estimate for the average observational error during tracking. Appendix B describes the changes necessary in performing least squares recursive smoothing over a fixed number of prior observations — in contrast to the method in the main text, where the smoothing is effective over all observations. Finally, in Appendix C, the extension of the method to include estimation of acceleration is outlined.

The derivation and discussion of this method of data smoothing are presented in the context of radar observations of moving bodies. However, this technique can be applied to any observable quantity which can be expressed as a linear combination of functions where the expansion coefficients are to be estimated.

II. CONSTANT DATA INTERVALS

The derivation of this method of performing recursive least squares smoothing is based on a technique employed by Kaplan.[2] For simplicity, we initially consider the case of straight-line unaccelerated flight. A sequence of observations $\mathring{x}_i$, $i = 1, 2, \cdots, n$, τ seconds apart, are made by an instrument whose measurement errors are taken to be uncorrelated

and normally distributed with mean zero and variance σ_i^2. We wish to obtain the least squares estimate of the position ($\bar{x}_n$) and velocity ($\bar{v}_n$) as of the nth observation* by suitably combining the latest observation ($\mathring{x}_n$) with a linear combination of the $n - 1$ previous observations. To do this, we will first minimize the sum of squared deviations from the least squares line of regression, R_n^2, assuming n observations have been made. We may write this sum as follows:

$$R_n^2 = \sum_{i=1}^{n} \{\mathring{x}_i - [\bar{x}_n - (n - i)\bar{u}_n]\}^2 w_i , \tag{1}$$

where $\bar{u}_n = \bar{v}_n \tau$ and w_i is an arbitrary weighting factor. Differentiating this equation with respect to $\bar{x}_n$ and $\bar{u}_n$ and setting each resulting equation equal to zero yields the normal equations:

$$\begin{aligned} \bar{x}_n \sum_{i=1}^{n} w_i - \bar{u}_n \sum_{i=1}^{n} (n - i)w_i &= \sum_{i=1}^{n} \mathring{x}_i w_i , \\ -\bar{x}_n \sum_{i=1}^{n} (n - i)w_i + \bar{u}_n \sum_{i=1}^{n} (n - i)^2 w_i &= -\sum_{i=1}^{n} \mathring{x}_i (n - i) w_i . \end{aligned} \tag{2}$$

We define the sums F_n, G_n, and H_n as

$$\begin{aligned} F_n &= \sum_{i=1}^{n} w_i , \\ G_n &= \sum_{i=1}^{n} (n - i) w_i , \\ H_n &= \sum_{i=1}^{n} (n - i)^2 w_i , \end{aligned} \tag{3}$$

so that equations (2) take the form

$$F_n \bar{x}_n - G_n \bar{u}_n = \sum_{i=1}^{n} \mathring{x}_i w_i , \tag{4}$$

$$-G_n \bar{x}_n + H_n \bar{u}_n = -\sum_{i=1}^{n} \mathring{x}_i (n - i) w_i . \tag{5}$$

The simultaneous solution of (4) and (5) yields the standard least squares estimates of position and velocity; however, they explicitly involve the presence of all observations $\mathring{x}_i$. To cast this in recursive form,

* Estimation to any other time $(n + p)\tau$ can be included by replacing (1) by

$$\sum_{i=1}^{n} \{\mathring{x}_i - [\bar{x}_{n+p} - (n + p - i)\bar{u}_{n+p}]\}^2 w_i = \min$$

we repeat the process of (1) through (5) omitting the last observation $\mathring{x}_n$. This yields estimates of the $(n - 1)$st position and velocity:

$$R_{n-1}^2 = \sum_{i=1}^{n-1} \{\mathring{x}_1 - [\bar{x}_{n-1} - (n - 1 - i)\bar{u}_{n-1}]\}^2 w_i . \tag{6}$$

Differentiating with respect to $\bar{x}_{n-1}$ and $\bar{u}_{n-1}$ and making use of definitions (3), the resulting equations may be written in the form

$$F_n(\bar{x}_{n-1} + \bar{u}_{n-1}) - G_n\bar{u}_{n-1} = \sum_{i=1}^{n-1} \mathring{x}_i w_i + w_n(\bar{x}_{n-1} + \bar{u}_{n-1}), \tag{7}$$

$$-G_n(\bar{x}_{n-1} + \bar{u}_{n-1}) + H_n\bar{u}_{n-1} = -\sum_{i=1}^{n} \mathring{x}_i(n - i)w_i. \tag{8}$$

Subtracting (7) from (4) and (8) from (5), we have

$$\begin{aligned} F_n[\bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})] - G_n(\bar{u}_n - \bar{u}_{n-1}) \\ = w_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})], \\ -G_n[\bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})] + H_n(\bar{u}_n - \bar{u}_{n-1}) = 0, \end{aligned} \tag{9}$$

whose solutions are:

$$\bar{x}_n = (\bar{x}_{n-1} + \bar{u}_{n-1}) + \alpha_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})], \tag{10}$$

$$\bar{u}_n = \bar{u}_{n-1} + \beta_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})], \tag{11}$$

where

$$\alpha_n = \frac{w_n H_n}{J_n}, \tag{12}$$

$$\beta_n = \frac{w_n G_n}{J_n}, \tag{13}$$

$$J_n = F_n H_n - G_n^2. \tag{14}$$

Equations (10) and (11) explicitly indicate that a smoothed coordinate is a linear combination of "old" data with a new observation; α_n and β_n are the position and velocity smoothing coefficients and are calculated for each new observation from (12) and (13) using definitions (3) and (14).

The estimates $(\bar{x}_{n-1} + \bar{u}_{n-1})$ and $\bar{u}_{n-1}$ are merely the predicted nth

position and velocity (times the data interval) based on $n - 1$ observations, so that we may define

$$\hat{x}_n = \bar{x}_{n-1} + \bar{u}_{n-1} , \qquad \hat{u}_n = \bar{u}_{n-1} . \tag{15}$$

The smoothing equations now take the form

$$\bar{x}_n = \hat{x}_n + \alpha_n(\mathring{x}_n - \hat{x}_n) = (1 - \alpha_n)\hat{x}_n + \alpha_n\mathring{x}_n , \tag{16}$$

$$\bar{u}_n = \hat{u}_n + \beta_n(\mathring{x}_n - \hat{x}_n) = (\hat{u}_n - \beta_n\hat{x}_n) + \beta_n\mathring{x}_n , \tag{17}$$

which show that the predicted nth position and velocity may be used to represent all past data.

The ability to optimize the smoothing for varying measurement errors σ_i^2 arises from the fact that the quantities F_n , G_n , H_n , and J_n can themselves be summed recursively:

$$\begin{aligned} F_n &= F_{n-1} + w_n , \\ G_n &= G_{n-1} + F_{n-1} , \\ H_n &= H_{n-1} + 2G_{n-1} + F_{n-1} , \\ J_n &= J_{n-1} + w_nH_n . \end{aligned} \tag{18}$$

Thus, each new observation can be arbitrarily weighted by w_n and, as can be shown by statistical analysis, optimumly weighted by choosing $w_n = 1/\sigma_n^2$. Since J_n is defined by (14), the last recursion relation above may be used as a consistency check. Note that a missing observation may be properly accounted for by choosing its weighting factor equal to zero and cycling the sums as usual. Equations (10) through (13) then set the smoothed coordinates equal to their predicted values, and the future weighting of both old and new data is now altered to compensate for the missing observation. To illustrate this, Tables I and II have been constructed to show how the weighting of data changes for the case of missing second and fifth observations. Here we have chosen $w_i = 1$ for simplicity. Note that the effect of the missing observations on the variance ratios $\sigma_{\bar{x}_n}^2/\sigma_0^2$ and $\sigma_{\bar{u}_n}^2/\sigma_0^2$ (see Table II) diminishes rapidly with the addition of new observations compared to the values of these ratios when all observations are present.

The actual weighting coefficients applied to the observations to yield the smoothed position and velocity coordinates can be determined by solving (4) and (5) simultaneously. If we define

$$\bar{x}_n = \sum_{i=1}^{n} c_i\mathring{x}_i , \qquad \bar{u}_n = \sum_{i=1}^{n} d_i\mathring{x}_i , \tag{19}$$

TABLE I — ALL OBSERVATIONS PRESENT ($w_i = 1$)

n	1	2	3	4	5	6	7	8
F	1	2	3	4	5	6	7	8
G	0	1	3	6	10	15	21	28
H	0	1	5	14	30	55	91	140
J	0	1	6	20	50	105	196	336
α	(1)	1	5/6	7/10	3/5	11/21	13/28	5/12
β	(0)	1	1/2	3/10	1/5	3/21	3/28	1/12
$\sigma_{\bar{x}}^2/\sigma_0^2$	1	1	0.833	0.700	0.600	0.524	0.464	0.417
$\sigma_{\bar{u}}^2/\sigma_0^2$		2	0.500	0.200	0.100	0.057	0.036	0.024
	$\mathring{x}_1$	$\mathring{x}_2$	$\mathring{x}_3$	$\mathring{x}_4$	$\mathring{x}_5$	$\mathring{x}_6$	$\mathring{x}_7$	$\mathring{x}_8$
$\bar{x}_1$	1							
$\bar{x}_2$	0	1						
$\bar{x}_3$	−1/6	2/6	5/6					
$\bar{x}_4$	−2/10	1/10	4/10	7/10				
$\bar{x}_5$	−1/5	0	1/5	2/5	3/5			
$\bar{x}_6$	−4/21	−1/21	2/21	5/21	8/21	11/21		
$\bar{x}_7$	−5/28	−2/28	1/28	4/28	7/28	10/28	13/28	
$\bar{x}_8$	−2/12	−1/12	0	1/12	2/12	3/12	4/12	5/12
$\bar{u}_1$								
$\bar{u}_2$	−1	1						
$\bar{u}_3$	−1/2	0	1/2					
$\bar{u}_4$	−3/10	−1/10	1/10	3/10				
$\bar{u}_5$	−2/10	−1/10	0	1/10	2/10			
$\bar{u}_6$	−5/35	−3/35	−1/35	1/35	3/35	5/35		
$\bar{u}_7$	−3/28	−2/28	−1/28	0	1/28	2/28	3/28	
$\bar{u}_8$	−7/84	−5/84	−3/84	−1/84	1/84	3/84	5/84	7/84

then the solution of (4) and (5) yields:

$$c_i = \frac{1}{J_n}[H_n - (n - i)G_n]w_i, \tag{20}$$

$$d_i = \frac{1}{J_n}[G_n - (n - i)F_n]w_i. \tag{21}$$

These coefficients are tabulated in Tables I and II for the special cases considered.

In digital computer applications, this method would require storage of $\hat{x}_n$, $\hat{u}_n$, F_{n-1}, G_{n-1}, H_{n-1} (and J_{n-1}, if desired). Upon receipt of $\mathring{x}_n$, w_n is determined; the sums are updated, (18); α_n and β_n are computed, (12) and (13); $\bar{x}_n$ and $\bar{u}_n$ are determined, (16) and (17); and $\hat{x}_{n+1}$ and $\hat{u}_{n+1}$ are formed, (15), and stored with the current values of the sums. The amount of storage and computation per cycle is independent of the number of observations.

In situations where this type of smoothing can be employed, the method outlined here offers several advantages over conventional

TABLE II — OBSERVATIONS 2 AND 5 MISSED

n	1	2	3	4	5	6	7	8
F	1	1	2	3	3	4	5	6
G	0	1	2	4	7	10	14	19
H	0	1	4	10	21	38	62	95
J	0	0	4	14	14	52	114	209
α	(1)	0	1	5/7	0	19/26	31/57	95/209
β	(0)	0	1/2	2/7	0	5/26	7/57	19/209
σ_x^2/σ_0^2	1		1	0.714	1.500	0.741	0.544	0.455
σ_u^2/σ_0^2			0.500	0.214	0.214	0.077	0.044	0.029
	$\mathring{x}_1$	$\mathring{x}_2$	$\mathring{x}_3$	$\mathring{x}_4$	$\mathring{x}_5$	$\mathring{x}_6$	$\mathring{x}_7$	$\mathring{x}_8$
$\bar{x}_1$	1							
$\bar{x}_2$								
$\bar{x}_3$	0		1					
$\bar{x}_4$	−1/7		3/7	5/7				
$\bar{x}_5$	−1/2		1/2	1				
$\bar{x}_6$	−6/26		4/26	9/26		19/26		
$\bar{x}_7$	−11/57		3/57	10/57		25/57	31/57	
$\bar{x}_8$	−38/209		0	19/209		57/209	76/209	95/209
$\bar{u}_1$								
$\bar{u}_2$								
$\bar{u}_3$	−1/2		1/2					
$\bar{u}_4$	−5/14		1/14	4/14				
$\bar{u}_5$	−5/14		1/14	4/14				
$\bar{u}_6$	−5/26		−1/26	1/26		5/26		
$\bar{u}_7$	−16/114		−6/114	−1/114		9/114	14/114	
$\bar{u}_8$	−23/209		−11/209	−5/209		7/209	13/209	19/209

methods. For example, the weighting factors w_i can be arbitrarily chosen without affecting the exactness condition; i.e., in the absence of errors, true straight-line data are unaltered by the smoothing process. Thus, the w_i's can be chosen to place greater (or lesser) emphasis on new data by increasing (or decreasing) their relative weight with respect to old data. If the measurement errors (σ_i^2) vary in a known way with position, velocity, or time, the w_i can be chosen equal to $1/\sigma_i^2$ so as to optimize the smoothing for these errors. It is important to note here that optimum smoothing requires only a knowledge of the ratio of the errors, not their magnitudes. Of course, the magnitudes must be known in order to evaluate the statistical accuracy of the smoothed coordinates.

The presence of the sums makes it possible to determine the statistical accuracy in the smoothed coordinates at any time (assuming uncorrelated Gaussian measurement errors) for two important cases:

(a) the weighting factors are chosen equal to K/σ_i^2, where K is a normalization constant;

(b) $\sigma_i^2 = \sigma_0^2$ (a constant for all i) and $w_i = 1$ or 0.

In hoth cases, we may write

$$\sigma_{\bar{x}_n}{}^2 = \sum_{i=1}^{n} c_i^2 \sigma_i^2, \qquad \sigma_{\bar{u}_n}{}^2 = \sum_{i=1}^{n} d_i^2 \sigma_i^2, \tag{22}$$

which, by making use of (3) and (14), can be evaluated as

$$\sigma_{\bar{x}_n}{}^2 = K \frac{H_n}{J_n}, \tag{23}$$

$$\sigma_{\bar{u}_n}{}^2 = K \frac{F_n}{J_n} \quad \left(\text{or } \sigma_{\bar{v}_n} = \frac{K}{\tau^2} \frac{F_n}{J_n}\right) \tag{24}$$

for case (a), and K replaced with σ_0^2 for case (b).

The accuracy of an arbitrary prediction of p units ($p\tau$ seconds) into the past or future may be determined from

$$\begin{aligned} \sigma_{\hat{x}_{n+p}}{}^2 &= \sum_{i=1}^{n} (c_i + pd_i)^2 \sigma_i^2, \\ &= \frac{K}{J_n} [H_n + 2pG_n + p^2 F_n]. \end{aligned} \tag{25}$$

Equation (25) is of great value in certain tracking systems where the $(n + 1)$st observation is "captured" by making a prediction, $\hat{x}_{n+1}$ of $\mathring{x}_{n+1}$ and surrounding it with a gate within which the observation must fall. The optimum gate size is obtained by adding the variances of the $(n + 1)$st observation and the prediction. Thus, using (25) with $p = 1$, we find:

$$\text{optimum 1-sigma gate} = \begin{cases} \sqrt{\sigma_{n+1}{}^2 + K \dfrac{H_{n+1}}{J_n}}, & \text{case (a)}, \\ \sigma_0 \sqrt{1 + \dfrac{H_{n+1}}{J_n}}, & \text{case } (b). \end{cases} \tag{26}$$

The gate size determined from (26) is automatically increased when observations are missed, and is optimum for any sequence of observations and misses.

For completeness, we present the explicit value of all quantities after n observations for the case of all $w_i = 1$:

$$F_n = n,$$

$$G_n = \frac{n(n-1)}{2},$$

$$H_n = \frac{n(n-1)(2n-1)}{6},$$

$$J_n = \frac{n^2(n^2-1)}{12},$$

$$\alpha_n = \frac{2(2n-1)}{n(n+1)},$$

$$\beta_n = \frac{6}{n(n+1)},$$

$$c_i = \frac{2[(2n-1) - 3(n-i)]}{n(n+1)},$$

$$d_i = \frac{6[(n-1) - 2(n-i)]}{(n-1)n(n+1)},$$

$$\sigma_{\bar{x}_n}{}^2 = \sigma_0{}^2 \frac{2(2n-1)}{n(n+1)} = \alpha_n \sigma_0{}^2,$$

$$\sigma_{\bar{v}_n}{}^2 = \frac{\sigma_0{}^2}{\tau^2} \frac{12}{(n-1)n(n+1)},$$

$$\sigma_{\hat{x}_{n+p}}{}^2 = 2\sigma_0{}^2 \frac{(n-1)(2n-1) + 6p(n-1) + 6p^2}{(n-1)n(n+1)}.$$

optimum

$$\text{1-sigma gate} = \sigma_0 \sqrt{\frac{(n+1)(n+2)}{n(n-1)}}.$$

III. VARIABLE DATA INTERVALS

In many cases of interest, the time intervals between observations may vary over wide limits. The results of the previous section can easily be extended to include the case of arbitrary observation times $\mathring{t}_i$ by replacing the quantity $(n - i)$ by $(\mathring{t}_n - \mathring{t}_i)$ and replacing u_n by v_n. The sums G_n and H_n are redefined as

$$G_n = \sum_{i=1}^{n} (\mathring{t}_n - \mathring{t}_i) w_i, \qquad (27)$$

$$H_n = \sum_{i=1}^{n} (\mathring{t}_n - \mathring{t}_i)^2 w_i,$$

where F_n is unchanged, (3), and the recursion relations become

$$\begin{aligned} F_n &= F_{n-1} + w_n , \\ G_n &= G_{n-1} + (\mathring{t}_n - \mathring{t}_{n-1})F_{n-1} , \\ H_n &= H_{n-1} + 2(\mathring{t}_n - \mathring{t}_{n-1})G_{n-1} + (\mathring{t}_n - \mathring{t}_{n-1})^2 F_{n-1} \\ &= H_{n-1} + (\mathring{t}_n - \mathring{t}_{n-1})(G_n + G_{n-1}), \\ J_n &= J_{n-1} + w_n H_n . \end{aligned} \tag{28}$$

The smoothing equations retain the same form, except that the smoothing coefficient β_n now implicitly contains the time dependence

$$\bar{v}_n = \hat{v}_n + \beta_n(\mathring{x}_n - \hat{x}_n). \tag{29}$$

The predicted $(n + 1)$st observation, $\hat{x}_{n+1}$, is now computed from

$$\hat{x}_{n+1} = \bar{x}_n + (\mathring{t}_{n+1} - \mathring{t}_n)\bar{v}_n \tag{30}$$

where, obviously, one must either know $\mathring{t}_{n+1}$ before receipt of the $(n + 1)$st observation or defer computation of $\hat{x}_{n+1}$ until after the $(n + 1)$st observation is made. For those systems which must predict the $(n + 1)$st observation in order to "capture" it, one must estimate the time of observation $(\hat{t}_{n+1})$ and apply an appropriate gate about $\hat{x}_{n+1}' = \bar{x}_n + (\hat{t}_{n+1} - \mathring{t}_n)\bar{v}_n$, large enough to account for all sources of error. No degradation in the quality of the fit is made if only a poor estimate of $\hat{t}_{n+1}$ can be obtained, since this is merely a device for capturing $\mathring{x}_{n+1}$. Once the observation has been received, $\hat{x}_{n+1}'$ can be corrected to yield $\hat{x}_{n+1}$ as follows:

$$\hat{x}_{n+1} = \hat{x}_{n+1}' + (\mathring{t}_{n+1} - \hat{t}_{n+1})\hat{v}_{n+1} . \tag{31}$$

In the preceding section, we described how a missing observation could be properly accounted for. In the case of variable data intervals, only the time difference between successive observations enters the equations. Thus, an observation that was anticipated ($\mathring{x}_{n+1}$ at $\mathring{t}_{n+1}$) but never made $(w_{n+1} = 0)$ has no effect on the equation. The next observation is predicted at time $\mathring{t}_{n+2}$, i.e.,

$$\begin{aligned} \hat{x}_{n+2} &= \hat{x}_{n+1} + (\mathring{t}_{n+2} - \mathring{t}_{n+1})\hat{v}_{n+1} \\ &= \bar{x}_n + (\mathring{t}_{n+2} - \mathring{t}_n)\bar{v}_n , \end{aligned} \tag{32}$$

and the only quantity entering the equations is $(\mathring{t}_{n+2} - \mathring{t}_n)$.

The determination of the optimum gate size is now somewhat more

complicated than before. In general, $\hat{t}_{n+1}$ is determined from some function of present position and velocity, $T(\bar{x}_n, \bar{v}_n)$, so that

$$\hat{x}_{n+1} = \bar{x}_n + \bar{v}_n T(\bar{x}_n, \bar{v}_n).$$

In the approximation for small estimation errors, $\sigma_{\hat{x}_{n+1}}{}^2$ can be determined from*

$$\sigma_{\hat{x}_{n+1}}{}^2 = \sigma_{\bar{x}_n}{}^2 \left(\frac{\partial \hat{x}_{n+1}}{\partial \bar{x}_n}\right)^2 + 2\sigma_{\bar{x}_n\bar{v}_n} \left(\frac{\partial \hat{x}_{n+1}}{\partial \bar{x}_n}\right)\left(\frac{\partial \hat{x}_{n+1}}{\partial \bar{v}_n}\right) + \sigma_{\bar{v}_n}{}^2 \left(\frac{\partial \hat{x}_{n+1}}{\partial \bar{v}_n}\right)^2, \quad (33)$$

where, using (19) through (22) and definition (14), the covariance may be evaluated as

$$\sigma_{\bar{x}_n\bar{v}_n} = \sum_{i=1}^{n} c_i d_i \sigma_i^2 = K \frac{G_n}{J_n}$$

and the gate size becomes

$$\text{optimum 1-sigma gate} = \sqrt{\sigma_{\hat{x}_{n+1}}{}^2 + \sigma_{\dot{x}_{n+1}}{}^2}.$$

For most practical purposes, an adequate approximation to the optimum gate size can be made by first estimating $\hat{t}_{n+1}$ and then evaluating

$$\hat{H}_{n+1} = H_n + 2(\hat{t}_{n+1} - \mathring{t}_n)G_n + (\hat{t}_{n+1} - \mathring{t}_n)^2 F_n, \quad (34)$$

which is inserted in (26).

Note that for this case of variable data intervals, the quantities H_n/J_n and F_n/J_n still determine the instantaneous position and velocity variances, respectively, and (25) (with p now equal to the prediction time in seconds) determines the statistical accuracy of an arbitrary extrapolation or interpolation.

IV. KNOWN ACCELERATION

The previous sections have dealt with the case of zero acceleration, which is but a special case of motion with known acceleration. For the case of known and constant acceleration a, all the preceding results apply if one modification of the smoothing equation is made. The predicted $(n + 1)$st position and velocity should be determined as follows:

$$\begin{aligned} \hat{x}_{n+1} &= \bar{x}_n + \bar{v}_n(\mathring{t}_{n+1} - \mathring{t}_n) + \tfrac{1}{2}a(\mathring{t}_{n+1} - \mathring{t}_n)^2, \\ \hat{v}_{n+1} &= \bar{v}_n + a(\mathring{t}_{n+1} - \mathring{t}_n) \end{aligned} \quad (35)$$

with $(\mathring{t}_{n+1} - \mathring{t}_n) = \tau$ for the case of constant data intervals.

* See, for example, Ref. 3, p. 51.

If the acceleration is a known function of the position and velocity $a(x, v)$, one could treat this case as above by evaluating $a(x, v)$ at $(\bar{x}_n, \bar{v}_n)$ and using (35). This procedure works very well if $a(x, v)$ is not a sensitive function of position and velocity and the time interval between observations is not too large. Possible criteria for these restrictions for the case of constant data intervals are

$$\frac{\tau^2}{2}[a(\bar{x}_n \pm \sigma_{\bar{x}_n}, \bar{v}_n \pm \sigma_{\bar{v}_n}) - a(\bar{x}_n, \bar{v}_n)] \ll \sigma_{x_{n+1}}$$

or

$$\tau[a(\bar{x}_n \pm \sigma_{\bar{x}_n}, \bar{v}_n \pm \sigma_{\bar{v}_n}) - a(\bar{x}_n, \bar{v}_n)] \ll \sigma_{\dot{v}_{n+1}},$$

which express the fact that the variation in the acceleration due to the errors in the smoothed data should not contribute significantly to the error in predicted position or velocity. For acceleration functions or smoothing times [i.e., $(n - 1)\tau$] which do not satisfy the above criteria, other methods must be employed to optimize the smoothing.

In order to assess the systematic error in smoothed position and velocity due to a varying component of acceleration not compensated for by the method outlined above, one can replace the varying component by some average value. Then it is easy to show that, for a constant acceleration a, the differences between true position and velocity and the linearly smoothed values are

$$\begin{aligned} x_n^T - \bar{x}_n &= \frac{(n-1)(n-2)}{6}\left(\frac{a\tau^2}{2}\right), \\ v_n^T - \bar{v}_n &= \frac{n-1}{2}(a\tau). \end{aligned} \tag{36}$$

It is obvious that this method of performing least squares smoothing can be extended to the case of an unknown but *constant* acceleration which is a special case of motion with known "jerk" (rate of change of acceleration). The derivation and some results for this case are given in Appendix C, and a comparison with the estimation errors for linear smoothing is made.

V. ACKNOWLEDGMENTS

To the extent of the author's knowledge, the method of performing least squares smoothing described here is new, but many of the results derived from the method are not. It has not been the author's purpose to present the results with mathematical rigor, but rather to indicate the

flexibility of this technique. For a more complete discussion of least squares smoothing and some interesting variants, the reader is referred to a partial list of references included at the end.[4,5,6,7,8]

The author wishes to thank G. L. Baldwin for a valuable discussion and R. B. Blackman for very helpful advice and comments.

APPENDIX A

*Recursive Summation of Squared Residuals**

In connection with a study of methods to detect errors in the smoothing of data by the technique described in the main text, it was found that the sum of squared deviations from the least squares line of regression could be determined recursively. Insofar as this may be of some interest in certain data processing systems, we will briefly outline the derivation and present the results in this appendix.

The sum of squared deviations from the least squares line of regression (hereafter called squared residuals) is defined by (1), which we repeat here:

$$R_n^2 = \sum_{i=1}^{n} [\mathring{x}_i - \bar{x}_n + (n - i)\bar{u}_n]^2 w_i \, . \tag{37}$$

The recursion relation may be obtained by replacing all n's by $(n + 1)$'s in (37):

$$R_{n+1}^2 = \sum_{i=1}^{n+1} [\mathring{x}_i - \bar{x}_{n+1} + (n + 1 - i)\bar{u}_{n+1}]^2 w_i \, , \tag{38}$$

which can be immediately converted to:

$$R_{n+1}^2 = \sum_{i=1}^{n} [\mathring{x}_i - \bar{x}_{n+1} + (n + 1 - i)\bar{u}_{n+1}]^2 w_i + w_{n+1}(\mathring{x}_{n+1} - \bar{x}_{n+1})^2.$$

By making use of the smoothing and prediction equations, (10), (11), and (15), we can rewrite the last result as

$$\begin{aligned} R_{n+1}^2 = \sum_{i=1}^{n} \big([\mathring{x}_i - \bar{x}_n + (n - i)\bar{u}_n] \\ - \{(\mathring{x}_{n+1} - \hat{x}_{n+1})[\alpha_{n+1} - (n + 1 - i)\beta_{n+1})]\}\big)^2 w_i \\ + w_{n+1}(\mathring{x}_{n+1} - \bar{x}_{n+1})^2. \end{aligned} \tag{39}$$

The evaluation of this sum is straightforward but tedious and makes

* See also Ref. 7.

use of the defining relations for the quantities F_n, G_n, H_n, J_n, α_n, and β_n, (3), (12), (13), (14), and the recursion relations, (18). The result may be written in either of two forms:

$$\begin{aligned} R_{n+1}^2 &= R_n^2 + \frac{w_{n+1}}{1 - \alpha_{n+1}} (\mathring{x}_{n+1} - \bar{x}_{n+1})^2 \\ &= R_n^2 + w_{n+1}(1 - \alpha_{n+1})(\mathring{x}_{n+1} - \hat{x}_{n+1})^2. \end{aligned} \tag{40}$$

Thus only one storage slot and a modest amount of computation need be invested to obtain the instantaneous value of the squared residuals.

In order to use this as an error-detecting device, some assumptions regarding the observational error must be made. If the variations in the σ_i^2 are known and compensated for by choosing $w_n = K/\sigma_n^2$, and if the errors are uncorrelated between observations, then the average value of R_n^2 can be shown to be

$$\text{ave } (R_n^2) = K[(\text{number of observations}) - 2]. \tag{41}$$

If the $\sigma_i^2 = \sigma_0^2$ and $w_i = 1$ or 0, then

$$\text{ave } (R_n^2) = \sigma_0^2(F_n - 2) \tag{42}$$

since, for this case, F_n is equal to the number of observations. The fact that the average value is proportional to two less than the number of observations is related to the fact that two degrees of freedom have been used up in determining $\bar{x}_n$ and $\bar{u}_n$. If the variations in σ_i^2 are unknown or too complicated to compensate for, then it is often possible, for a given system, to determine some bounds on the average value of squared residuals.

If we define

$$\bar{R}_n^2 = \frac{R_n^2}{(\text{number of observations}) - 2}$$

then, from above we have

$$\text{ave } (\bar{R}_n^2) = \begin{cases} K & \text{for } w_i = K/\sigma_i^2, \\ \sigma_0^2 & \text{for } w_i = 1 \text{ or } 0. \end{cases} \tag{43}$$

It can be shown from a straightforward application of statistical analysis* that

$$\text{var } (\bar{R}_n^2) = \frac{2K^2}{(\text{number of observations}) - 2} \tag{44}$$

with K^2 replaced by σ_0^4 for the case of $w_i = 1$ or 0.

* See, for example, Ref. 3, p. 103.

If no prior knowledge of the magnitude of the measurement errors is available, the squared residuals can be used to obtain an estimate of the average measurement error if all $w_i = 1$, or of the average *effective* measurement error if an arbitrary weighting sequence is used.

APPENDIX B

Fixed Memory Smoothing[5]

In contrast to the method described in the main text, where the smoothing is designed to include all observations (i.e., variable smoothing time), we can set up a smoothing procedure which fits a least squares line of regression to the latest r observations. The ability to optimize for varying observational errors and missing observations is retained, but we must provide storage for the r observations.

The two quantities to be minimized may be written as

$$\begin{gathered} \sum_{i=n+1-r}^{n} [\mathring{x}_i - \bar{x}_n + (n - i)\bar{u}_n]^2 w_i \,; \\ \sum_{i=n-r}^{n-1} [\mathring{x}_i - \hat{x}_n + (n - i)\hat{u}_n]^2 w_i \,. \end{gathered} \tag{45}$$

We define the sums F_n, G_n, H_n, and J_n as follows:

$$\begin{aligned} F_n &= \sum_{i=n+1-r}^{n} w_i \,, \\ G_n &= \sum_{i=n+1-r}^{n} (n - i) w_i \,, \\ H_n &= \sum_{i=n+1-r}^{n} (n - i)^2 w_i \,, \\ J_n &= F_n H_n - G_n^2, \end{aligned} \tag{46}$$

and their recursion relations are

$$\begin{aligned} F_n &= F_{n-1} + w_n - w_{n-r} \,, \\ G_n &= G_{n-1} + F_{n-1} - r w_{n-r} \,, \\ H_n &= H_{n-1} + 2G_{n-1} + F_{n-1} - r^2 w_{n-r} \,, \\ J_n &= J_{n-1} + w_n H_n - w_{n-r}[H_{n-1} - 2(r - 1)G_{n-1} + (r - 1)^2 F_{n-1}]. \end{aligned} \tag{47}$$

The smoothing equations can be shown to be

$$\begin{aligned} \bar{x}_n &= \hat{x}_n + \alpha_n(\mathring{x}_n - \hat{x}_n) + \gamma_n[\mathring{x}_{n-r} - (\hat{x}_n - r\hat{u}_n)], \\ \bar{u}_n &= \hat{u}_n + \beta_n(\mathring{x}_n - \hat{x}_n) + \delta_n[\mathring{x}_{n-r} - (\hat{x}_n - r\hat{u}_n)], \end{aligned} \tag{48}$$

where

$$\begin{aligned} \alpha_n &= \frac{w_n H_n}{J_n}, \qquad \gamma_n = \frac{w_{n-r}(rG_n - H_n)}{J_n}, \\ \beta_n &= \frac{w_n G_n}{J_n}, \qquad \delta_n = \frac{w_{n-r}(rF_n - G_n)}{J_n}, \end{aligned} \tag{49}$$

and the prediction equations are the same as (15).

The quantities $(rG_n - H_n)$ and $(rF_n - G_n)$ may also be written recursively, for if we define

$$A_n = rF_n - G_n,$$

$$B_n = rG_n - H_n,$$

then

$$A_n = A_{n-1} - F_{n-1} + rw_{n-r},$$

$$B_n = B_{n-1} + A_{n-1} - (G_{n-1} + F_{n-1}).$$

The explicit appearance of $\mathring{x}_{n-r}$ in (48) demonstrates the fact that storage must be provided for the last r observations. As in the method described in the main text, the w_i can be chosen completely arbitrarily and if they are chosen equal to $1/\sigma_n^2$, the smoothing is optimized for the varying data error. It may be advantageous to provide storage for the r weighting factors w_i if they are computed from some function. This would remove the necessity for recomputing w_{n-r} when it had already been computed at a previous $n' = n - r$.

In terms of computer operation, the time required to process each new observation is independent of r. Thus significant savings in computation time over other methods (stored coefficients, cascaded simple sums) are achieved only when r is large. However, these other methods smooth data in a predetermined manner and cannot alter the weighting of data to compensate for an arbitrary sequence of observations and misses or optimize for trajectory-dependent measurement errors.

For the case of all $w_i = 1$, the explicit values of all quantities are given at the end of Section II of the main text, with n replaced by r.

APPENDIX C

Estimation of Acceleration

To distinguish between smoothing over observations of linear and parabolic flight, we shall refer to the former as linear smoothing and the latter as quadratic smoothing. For the latter, we want to minimize

$$\sum_{i=1}^{n} \{\mathring{x}_i - [\bar{x}_n - (n - i)\bar{u}_n + (n - i)^2\bar{s}_n]\}^2 w_i , \qquad (50)$$

where

$$\bar{u}_n = \bar{v}_n\tau, \qquad \bar{s}_n = \tfrac{1}{2}\bar{a}_n\tau^2.$$

Straightforward differentiation of this expression with respect to the three unknowns yields the normal equations:

$$\begin{aligned} F_n\bar{x}_n - G_n\bar{u}_n + H_n\bar{s}_n &= \sum_{i=1}^{n} \mathring{x}_i w_i , \\ -G_n\bar{x}_n + H_n\bar{u}_n - I_n\bar{s}_n &= -\sum_{i=1}^{n} \mathring{x}_i(n - i)w_i , \\ H_n\bar{x}_n - I_n\bar{u}_n + K_n\bar{s}_n &= \sum_{i=1}^{n} \mathring{x}_i(n - i)^2 w_i , \end{aligned} \qquad (51)$$

where F_n, G_n, and H_n are defined by (3) and

$$\begin{aligned} I_n &= \sum_{i=1}^{n} (n - i)^3 w_i , \\ K_n &= \sum_{i=1}^{n} (n - i)^4 w_i . \end{aligned} \qquad (52)$$

The recursion relations for I_n and K_n are

$$I_n = I_{n-1} + 3H_{n-1} + 3G_{n-1} + F_{n-1} ,$$

$$K_n = K_{n-1} + 4I_{n-1} + 6H_{n-1} + 4G_{n-1} + G_{n-1} ,$$

and, as before, J_n is defined as the determinant of the coefficients in the normal equations:

$$J_n = F_nH_nK_n - 2G_nH_nI_n - H_n^3 - F_nI_n^2 - K_nG_n^2. \qquad (53)$$

Repeating the procedure for the estimates $\bar{x}_{n-1}$, $\bar{u}_{n-1}$, and $\bar{s}_{n-1}$ and subtracting the resulting equations from the corresponding equations in

(51), we obtain the three-variable equivalent to (9). The solution for quadratic smoothing can be written

$$
\begin{aligned}
\bar{x}_n &= \hat{x}_n + \alpha_n(\mathring{x}_n - \bar{x}_n), \\
\bar{u}_n &= \hat{u}_n + \beta_n(\mathring{x}_n - \hat{x}_n), \\
\bar{s}_n &= \hat{s}_n + \gamma_n(\mathring{x}_n - \hat{x}_n),
\end{aligned} \tag{54}
$$

where

$$
\begin{aligned}
\alpha_n &= w_n \frac{H_n K_n - I_n^2}{J_n}, \\
\beta_n &= w_n \frac{G_n K_n - H_n I_n}{J_n}, \\
\gamma_n &= w_n \frac{G_n I_n - H_n^2}{J_n}.
\end{aligned} \tag{55}
$$

The auxiliary prediction equations are

$$
\begin{aligned}
\hat{x}_{n+1} &= \bar{x}_n + \bar{u}_n + \bar{s}_n , \\
\hat{u}_{n+1} &= \bar{u}_n + 2\bar{s}_n , \\
\hat{s}_{n+1} &= \bar{s}_n ,
\end{aligned} \tag{56}
$$

where additional terms may be added to compensate for known "jerk."

If, in addition to the definitions (19), we define

$$
s_n = \sum_{i=1}^{n} e_i \mathring{x}_i ,
$$

the weighting coefficients c_i, d_i, and e_i applied to the observations to yield smoothed position, velocity and acceleration are:

$$
\begin{aligned}
c_i &= \frac{1}{J_n}[(H_n K_n - I_n^2) - (G_n K_n - H_n I_n)(n - i) \\
&\qquad + (G_n I_n - H_n^2)(n - i)^2]w_i, \\
d_i &= \frac{1}{J_n}[(G_n K_n - H_n I_n) - (F_n K_n - H_n^2)(n - i) \\
&\qquad + (F_n I_n - G_n H_n)(n - i)^2]w_i, \\
e_i &= \frac{1}{J_n}[(G_n I_n - H_n^2) - (F_n I_n - G_n H_n)(n - i) \\
&\qquad + (F_n H_n - G_n^2)(n - i)^2]w_i.
\end{aligned} \tag{57}
$$

The variance ratios for smoothed position, velocity, and acceleration (assuming constant observational error) can be evaluated as

$$\begin{aligned}
\sigma_{\bar{x}_n}^2/\sigma_0^2 &= \frac{H_n K_n - I_n^2}{J_n}, \\
\sigma_{\bar{u}_n}^2/\sigma_0^2 &= \frac{F_n K_n - H_n^2}{J_n}, \\
\sigma_{\bar{s}_n}^2/\sigma_0^2 &= \frac{F_n H_n - G_n^2}{J_n}.
\end{aligned} \tag{58}$$

For all $w_i = 1$, these quantities can be explicitly evaluated as functions of n:

$$F_n = n,$$

$$G_n = \frac{n(n-1)}{2},$$

$$H_n = \frac{n(n-1)(2n-1)}{6},$$

$$I_n = \frac{n^2(n-1)^2}{4},$$

$$K_n = \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30},$$

$$J_n = \frac{n^3(n-1)^2(n+1)^2(n-2)(n+2)}{2160},$$

$$\alpha_n = \frac{3(3n^2-3n+2)}{n(n+1)(n+2)},$$

$$\beta_n = \frac{18(2n-1)}{n(n+1)(n+2)},$$

$$\gamma_n = \frac{30}{n(n+1)(n+2)},$$

$$c_i = 3\,\frac{(3n^2-3n+2) - 6(2n-1)(n-i) + 10(n-i)^2}{n(n+1)(n+2)},$$

$$d_i = 6\,\frac{3(2n-1)(n-1)(n-2) - 2(8n-11)(2n-1)(n-i) + 30(n-1)(n-i)^2}{(n-2)(n-1)n(n+1)(n+2)},$$

$$e_i = 30\,\frac{(n-1)(n-2) - 6(n-1)(n-i) + 6(n-i)^2}{(n-2)(n-1)n(n+1)(n+2)},$$

$$\sigma_{\bar{x}_n}{}^2 = \sigma_0{}^2\left[\frac{3(3n^2-3n+2)}{n(n+1)(n+2)}\right] = \alpha_n\sigma_0{}^2,$$

$$\sigma_{\bar{v}_n}{}^2 = \frac{\sigma_0{}^2}{\tau^2}\left[\frac{12(8n-11)(2n-1)}{(n-2)(n-1)n(n+1)(n+2)}\right],$$

$$\sigma_{\bar{a}_n}{}^2 = \frac{4\sigma_0{}^2}{\tau^4}\left[\frac{180}{(n-2)(n-1)n(n+1)(n+2)}\right].$$

It is interesting to compare the position and velocity variance ratios for linear and quadratic smoothing:

$$\frac{\text{position variance (quadratic)}}{\text{position variance (linear)}} = \frac{3(3n^2-3n+2)}{2(n+2)(2n-1)} \to \frac{9}{4}\text{ for large } n,$$

$$\frac{\text{velocity variance (quadratic)}}{\text{velocity variance (linear)}} = \frac{(8n-11)(2n-1)}{(n-2)(n+2)} \to 16 \text{ for large } n.$$

These ratios indicate the cost in position and velocity accuracy for the inclusion of acceleration estimation.

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